CHAPTER-4

WAVE EQUATIONS

Structure

4.5 Wave Equation

The homogeneous Wave equation is

$$u_{tt} - \Delta u = 0 \qquad \dots (1)$$

and the non-homogeneous Wave equation

$$u_{tt} - \Delta u = f \qquad \dots (2)$$

Here $t > 0$ and $x \in U$, where $U \subset R^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to R, u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, ..., x_n)$. In equation (2) the function $f : U \times [0, \infty) \to R$ is given.

Remarks: 1. The Wave equation is a simplified model equation for a vibrating string (*n=1*). For *n=2,* it is membrane and it becomes an elastic solid for *n=3. u(x,t)* represents the displacement in some direction of the point *x* at time $t \geq 0$ for different values of *n*.

2. From physical perspective, it is obvious that we need initial condition on the displacement and velocity at time *t=0.*

Solution of Wave equation by spherical means (for n=1)

Theorem: Derive the solution of the initial value problem for one-dimensional Wave equation

$$u_{tt} - u_{xx} = 0 \text{ in } R \times (0, \infty) \qquad \dots (1)$$

$$u = g, u_t = h \text{ on } R \times \{t = 0\} \qquad \dots (2)$$

where g, h are given at time t=0..

Proof: The PDE (1) can be factored as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0 \qquad \dots (3)$$

Set

$$v(x,t)=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u(x,t) \qquad \ldots (4)$$

Then, equation (4) becomes

$$v_t(x,t)+v_x(x,t)=0 \qquad (x\in R, t>0) \qquad \ldots (5)$$

Equation (5) becomes the transport equation with constant coefficient (b=1).

Let $$v(x,0)=a(x) \qquad \ldots (6)$$

We know that the fundamental solution of the initial-value problem consisting of transport equation (5) and condition (6) is

$$v(x,t)=a(x-t), x\in R, t\geq 0 \qquad \ldots (7)$$

Combining equation (4) and (7), we obtain

$$u_t(x,t)-u_x(x,t)=a(x-t) \text{ in } R\times(0,\infty) \qquad \ldots (8)$$

Also

$$u(x,0)=g(x) \text{ in } R \qquad \ldots (9)$$

By virtue of initial condition (2), Equations (8) and (9) constitute the non-homogeneous transport problem. Hence its solution is

$$u(x,t)=g(x+t)+\int_0^t a\big(x+(s-t)(-1)-s\big)ds$$

$$=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} a(y)dy \qquad \ldots (10) \qquad (\because x+t-2s=y)$$

The second initial condition in (2) imply

$$a(x)=v(x,0)$$

$$=u_t(x,0)-u_x(0,0)$$

$$=h(x)-g'(x), x\in R \qquad \ldots (11)$$

Substituting (11) into (10)

$$u(x,t)=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t}\left[h(y)-g'(y)\right]dy$$

$$=\frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy \qquad \ldots (12)$$

for $x\in R, t\geq 0$.

This is the d' Alembert's formula.

Application of d' Alembert's Formula

Initial/boundary-value problem on the half line $R_+ = \{x > 0\}$.

Example: Consider the problem

$$\begin{cases} u_{tt} - u_{xx} & in & R_+ \times (0,\infty) \\ u = g, \quad u_t = h & on & R_+ \times \{t = 0\} \\ u = 0 & on & \{x = 0\} \times (0,\infty) \end{cases} \quad \text{...(1)}$$

where g, h are given, with

$$g(0) = 0, h(0) = 0. \quad \dots (2)$$

Solution: Firstly, we convert the given problems on the half-line into the problem on whole of R We do so by extending the functions u, g, h to all of R by odd reflection method as below we set.

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & for\ x \ge 0, t \ge 0 \\ -u(-x,t) & for\ x \le 0, t \ge 0 \end{cases} \quad \dots (3)$$

$$\tilde{g}(x) = \begin{cases} g(x) & for\ x \ge 0 \\ -g(x) & for\ x \le 0 \end{cases} \quad \dots (4)$$

$$\tilde{h}(x) = \begin{cases} h(x) & for\ x \ge 0 \\ -h(-x) & for\ x \le 0 \end{cases} \quad \dots (5)$$

Now, problem (1) becomes

$$\left.\begin{aligned} \tilde{u}_{tt} = \tilde{u}_{xx} \quad in\ R \times (0,\infty) \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h}\ on\ R \times \{t = 0\} \end{aligned}\right\} \quad \dots (6)$$

Hence, d' Alembert's formula for one-dimensional problem (6) implies

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \quad \dots (7)$$

Recalling the definition of $\tilde{u}, \tilde{g}, \tilde{h}$ in equations (3)-(5), we can transform equation (7) to read for $x \ge 0, t \ge 0$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & if\ x \ge t \ge 0 \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & if\ 0 \le x \le t \end{cases} \quad \dots (8)$$

Formula (8) is the solution of the given problem on the half-line $R_+ = \{x > 0\}$.

Solution of Wave Equation (for n=3)

Theorem: Derive Kirchhoff's formula for the solution of three-dimensional (n=3) initial-value problem

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad R^3 \times (0,\infty) \qquad \ldots(1)$$

$$u = g \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(2)$$

$$u_t = h \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(3)$$

Solution: Let us assume that $u \in C^2\left(R^3 \times [0,\infty)\right)$ solves the above initial-value problem.

As we know

$$U(x;r,t) = \oint_{\partial B(x,r)} u(y,t)\,ds(y) \qquad \ldots(4)$$

defines the average of $u(.,t)$ over the sphere $\partial B(x,r)$. Similarly,

$$G(x;r) = \oint_{\partial B(x,r)} g(y)\,ds(y) \qquad \ldots(5)$$

$$H(x;r) = \oint_{\partial B(x,r)} h(y)\,ds(y) \qquad \ldots(6)$$

We here after regard U as a function of r and t only for fixed x.

Next, set

$$\tilde{U} = rU\,, \qquad \ldots(7)$$

$$\tilde{G} = rG, \tilde{H} = rH \qquad \ldots(8)$$

We now assert that $\tilde{U}$ solve

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \; in & R_+ \times (0,\infty) \\ \tilde{U} = \tilde{G} \quad on & R_+ \times \{t=0\} \\ \tilde{U}_t = \tilde{H} \quad on & R_+ \times \{t=0\} \\ \tilde{U} = 0 \quad on & \{r=0\} \times (0,\infty) \end{cases} \qquad \ldots(9)$$

We note that the transformation in (7) and (8) convert the three-dimensional Wave equation into the one-dimensional Wave equation.

From equation (7)

$$\tilde{U}_{tt} = rU_{tt}$$

$$= r\left[U_{rr} + \frac{2}{r}U_r\right], \text{ Laplacian for n=3}$$

$$= rU_{rr} + 2U_r$$

$$= \left(U + rU_r\right)_r$$

$$= \left(\tilde{U}_r\right)_r = \tilde{U}_{rr} \qquad \ldots (10)$$

The problem (9) is one the half-line $R_+ = \{r \geq 0\}$.

The d' Alembert's formula for the same, for $0 \leq r \leq t$, is

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy \qquad \ldots (11)$$

From (4), we find

$$u(x,t) = \lim_{r \to 0^+} U(x;r,t) \qquad \ldots (12)$$

Equations (7),(8),(11) and (12) implies that

$$u(x,t) = \lim_{r \to 0^+}\left[\frac{\tilde{U}(x;r,t)}{r}\right]$$

$$= \lim_{r \to 0^+}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy\right]$$

$$= \tilde{G}'(t) + \tilde{H}(t) \qquad \ldots (13)$$

Owing then to (13), we deduce

$$u(x,t) = \frac{\partial}{\partial t}\left\{t \oint_{\partial B(x,t)} g(y)\,ds(y)\right\} + \left\{t \oint_{\partial B(x,t)} h(y)\,ds(y)\right\} \qquad \ldots (14)$$

But

$$\int_{\partial B(x,t)} g(y)\,ds(y) = \int_{\partial B(0,1)} g(x+tz)\,ds(z) \qquad \ldots (15)$$

Hence

$$\frac{\partial}{\partial t}\left\{\oint_{\partial B(x,t)} g(y)\,ds(y)\right\} = \oint_{\partial B(0,1)} \{Dg(x+tz)\}.z\,ds(z)$$

$$= \oint_{\partial B(x,t)} Dg(y).\left(\frac{y-x}{t}\right)ds(y) \qquad \ldots (16)$$

Now equation (14) and (16) conclude

$$u(x,t) = \oint_{\partial B(x,t)} \left[g(y) + \{Dg(y)\}.(y-x) + th(y) \right] ds(y) \qquad (17)$$

for $x \in R^3, t > 0$.

The formula (17) is called **KIRCHHOFF'S** formula for the solution of the initial value problem (1)-(3) in 3D.

4.6 Non-Homogeneous Problem

Now we investigate the initial-value problem for the non-homogeneous Wave equation

$$\begin{cases} u_{tt} - \Delta u = f \;\; in \;\; R^n \times (0,\infty) \\ u = 0, u_t = 0 \; on \; R^n \times \{t=0\} \end{cases} \qquad \ldots (1)$$

Motivated by Duhamel's principle, which says that one can think of the inhomogeneous problem as a set of homogeneous problems each starting afresh at a different time slice $t = t_0$. By linearity, one can add up (integrate) the resulting solutions through time t_0 and obtain the solution for the inhomogeneous problem. Assume that $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(.,s) - \Delta u(.,s) = 0 \quad in \;\; R^n \times (s,\infty) \\ u(.,s) = 0, u_t(.,s) = f(.,s) \, on \, R^n \times \{t=s\} \end{cases} \qquad \ldots (2)$$

and set

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad \left(x \in R^n, t \geq 0\right) \qquad \ldots (3)$$

Duhamel's principle asserts that this is solution of equation (1).

Theorem: Solution of Non-homogeneous Wave Equation

Let us consider the non-homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f \;\; in \;\; R^n \times (0,\infty) \\ u = 0, u_t = 0 \; on \; R^n \times \{t=0\} \end{cases} \qquad \ldots (1)$$

$f \in C^{\left[\frac{n}{2}\right]+1}\left(R^n \times [0,\infty)\right)$ and $n \geq 2$. Define u as

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad \left(x \in R^n, t \geq 0\right) \qquad \ldots (2)$$

Then

(i) $u \in C^2\left(R^n \times [0,\infty)\right)$

(ii) $u_{tt} - \Delta u = f$ in $R^n \times (0,\infty)$

(iii) $\lim\limits_{(x,t)\to(x^0,0)} u(x,t) = 0, \lim\limits_{(x,t)\to(x^0,0)} u_t(x,t) = 0$ for each point $x^0 \in R^n$ $(x \in R^n, t > 0)$.

Proof: (i) If n is odd, $\left[\frac{n}{2}\right]+1 = \frac{n+1}{2}$ and if n is even , $\left[\frac{n}{2}\right]+1 = \frac{n+2}{2}$

Also $u(.,.;s) \in C^2(R^n \times [s,\infty))$ for each $s \geq 0$ and so $u \in C^2(R^n \times [0,\infty))$.

Hence $u \in C^2(R^n \times [0,\infty))$.

(ii) Differentiating u w.r.t t and x by two times, we have

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds$$

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds$$

Furthermore,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)\,ds = \int_0^t u_{tt}(x,t;s)\,ds$$

Thus,

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t) \qquad x \in R^n, t \geq 0$$

(iii) And clearly $u(x,0) = u_t(x,0) = 0$ for $x \in R^n$. Therefore equation (2) is the solution of equation (1).

Examples: Let us work out explicitly how to solve (1) for n=1. In this case, d' Alembert's formula gives

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$$

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dyds$$

i.e. $$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds \qquad (x \in R, t \geq 0) \qquad \dots (5)$$

For n=3, Kirchhoff's formula implies

$$u(x,t;s) = (t-s) \oint_{\partial B(x,t-s)} f(y,s)\, dS$$

So that

$$u(x,t) = \int_0^t (t-s) \left(\oint_{\partial B(x,t-s)} f(y,s)\, dS \right) ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dS ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS dr$$

Therefore,

$$u(x,t) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy \qquad (x \in R^3, t \geq 0)$$

solves (4) for n=3.

The integrand on the right is called a retarded potential.

4.7 Energy Methods

There is the necessity of making more and more smoothness assumptions upon the data g and h to ensure the existence of a C^2 solution of the Wave equation for large and large n. This suggests that perhaps some other way of measuring the size and smoothness of functions may be more appropriate.

(a) Uniqueness

Let $U \subset R^n$ be a bounded, open set with a smooth boundary ∂U, and as usual set $U_T = U \times (0,T], \Gamma_T = \bar{U}_T - U_T$, where T>0. We are interested in the initial/boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f & in \quad U_T \\ u = g & on \quad \Gamma_T \\ u_t = h & on U \times \{t=0\} \end{cases} \qquad \ldots (1)$$

Theorem: There exists at most one function $u \in C^2(\bar{U}_T)$ solving (1).

Proof: If $\tilde{u}$ is another such solution, then $w := u - \tilde{u}$ solves

$$\begin{cases} w_{tt} - \Delta w = 0 & in \quad U_T \\ w = 0 & on \quad \Gamma_T \\ w_t = 0 & on U \times \{t=0\} \end{cases}$$

Set "energy"

$$e(t)=\frac{1}{2}\int_U w_t^2(x,t)+|Dw(x,t)|^2\,dx \qquad (0\le t\le T)$$

Differentiating e(t), we have

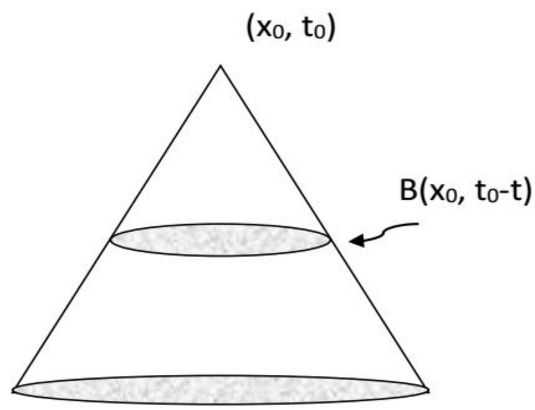
$$\dot{e}(t)=\int_U w_t w_{tt}+Dw.Dw_t dx$$

$$=\int_U w_t(w_{tt}-\Delta w)\,dx=0$$

There is no boundary term since w=0, and hence $w_t=0$, on $\partial U\times[0,T]$. Thus for all $0\le t\le T, e(t)=e(0)=0,$ and so $w_t, Dw=0$ within U_T. Since $w\equiv 0$ on $U\times\{t=0\}$, we conclude $w=u-\tilde{u}=0$ in U_T.

(b) Domain of Dependence

As another illustration of energy methods, let us examine again the domain of dependence of solutions to the Wave equation in all of space.



Cone of dependence

For this, suppose $u\in C^2$ solves

$$u_{tt}-\Delta u=0 \text{ in } R^n\times(0,\infty)$$

Fix $x_0\in R^n, t_0>0$ and consider the cone

$$C=\{(x,t)\,|\,0\le t\le t_0, |x-x_0|\le t_0-t\}.$$